Facets and Levels of Mathematical Abstraction

Hourya Benis Sinaceur

Draft / Paru dans :
Philosophia Scientiæ, vol.18, n°1, 2014, pp.81-12

Facets and Levels of Mathematical Abstraction*

Hourya Benis Sinaceur
IHPST–CNRS-Université Paris 1–ENS Ulm (France)

Résumé : L'abstraction mathématique consiste en la considération et la manipulation d'opérations, règles et concepts indépendamment du contenu dont les nantissent des applications particulières et du rapport qu'ils peuvent avoir avec les phénomènes et les circonstances du monde réel. L'abstraction mathématique emprunte diverses voies. Le terme « abstraction » ne désigne pas une procédure unique, mais un processus général où s'entrecroisent divers procédés employés successivement ou simultanément. En particulier, l'abstraction mathématique ne se réduit pas à la subsomption logique. Je vais étudier comparativement en quels termes les philosophes expliquent l'abstraction et par quels moyens les mathématiciens la mettent en œuvre. Je voudrais par là mettre en lumière les principaux processus de pensée en jeu et illustrer par des exemples divers niveaux d'intrication de techniques mathématiques récurrentes, qui incluent notamment la méthode axiomatique, les principes d'invariance, les relations d'équivalence et les correspondances fonctionnelles.

Abstract: Mathematical abstraction is the process of considering and manipulating operations, rules, methods and concepts divested from their reference to real world phenomena and circumstances, and also deprived from the content connected to particular applications. There is no one single way of performing mathematical abstraction. The term "abstraction" does not name a unique procedure but a general process, which goes many ways that are mostly simultaneous and intertwined; in particular, the process does not amount only to logical subsumption. I will consider comparatively how philosophers consider abstraction and how mathematicians perform it, with the aim to bring to light the fundamental thinking processes at play, and to illustrate by significant examples how much intricate and multi-leveled may be the combination of

*. I thank cordially the reviewers for their helpful comments to the previous version of this paper. I am indebted to them for many improvements. I thank also Jean-Pierre Marquis for discussions when he was visiting the University Paris-Diderot in 2012.

typical mathematical techniques which include axiomatic method, invariance principles, equivalence relations and functional correspondences.

Introduction

Mathematical abstraction is the process of considering and manipulating operations, rules, methods and concepts divested from their reference to real world phenomena and circumstances, and also deprived from the content connected to particular applications. So the abstract concept of number does not come down to any real aggregate (of sheep, beans, pencils, etc.) nor to any conceived collection (of geometrical points, numerical elements, unspecified elements, etc.) and it includes sets of numbers with rules of calculation different from the usual ones, such as the rules for quaternions, octonions, etc.

Actually, in mathematics, one encounters from the very beginning not one but several abstraction processes, which constitute specific and permanent ways of developing the mathematical core. In modern times, especially from the 19$^{\text{th}}$ century onwards, abstraction flourishes, and various processes are *more systematically* piled up, concatenated, and blended for producing procedures, entities, structures, and theories at higher and higher levels of abstraction. Abstracting is an ongoing *innovation* processing, which expands the mathematical stuff and makes it still richer and more and more intricate and layered.

I am not aiming at tackling head-on the fundamental question: "What is an abstract object?" or "In which sense abstract objects 'exist'?"[1] My purpose is much more modest and my method is mainly descriptive. I want to establish a picture of different and recurring procedures of mathematical abstraction. Thus, I will focus on different features of mathematical practice while I will disregard (explicit or implicit) ontological stands about the nature of mathematics and the status of its abstract objects. My purpose is epistemological, and it concerns the actual ways of performing abstraction in mathematical doing.[2] On the way I shall inevitably display how I see the means and products of mathematical activity. I think that focussing on actual mathematical abstraction processes may afford a *positive* picture of what is mathematical abstraction. I mean that we may come across criteria for being abstract that are not obtained by the classical "way of negation", an abstract object being

1. For discussing such questions belonging to "the *heroic* tradition in the philosophy of mathematics" [Kreisel 1985], see among others [Burgess & Rosen 1997], [Zalta 1983], [Rosen 2012], [Parsons 2008].

2. My approach seems to be similar to Jean-Pierre Marquis' approach in [Marquis forthcoming]. J.-P. Marquis makes fine distinctions between "symbolic", "formal", and "abstract", and also between "abstraction" and "generalization".

not located in space and time and *not* causally active.[3] I am rather taking "the way of example" despite its limits.

I will begin by a rapid incursion into the philosophical corpus. As a method I will focus on how the terms "abstract" and "abstraction" have been and are used; perforce I shall get information about the terms "concept" and "conceptualization", thanks to which I can make precise my understanding of mathematical concepts. I will then try to parallel the outcome of my inquiry with specific mathematical techniques: as a result we will notice that mathematical abstraction is not reducible to logical abstraction, at least as it was understood in the Aristotelian tradition. Thirdly, I attempt to describe the fundamental thinking processes underlying the main ways to get and increase abstraction, with a special attention to recurrent mathematical actions that produce more and more abstract objects. As a specific illustration, I am giving in a fourth section significant or emblematic examples; relying on them I want to stress that in mathematical practice several abstracting processes work simultaneously and interact together, conceptualization and axiomatization being an important *but only one* factor in the job of *systematic* and *uniform* problem solving.

1 Philosophical background

Philosophers may have recourse to mathematical practice and history of mathematics for making more precise and more substantial the understanding of some fundamental thought processes, such as *abstracting*. A philosophical mean at hand is to focus on the changes in meaning of the terms "abstraction" and "abstract". Such a semantic analysis provides indeed a crucial basis for contemporary linguistic, cultural, and conceptual understanding; it is largely used in "conceptual history", which may be internal (considering the rational links between mathematical concepts and methods) or external, considering the institutional, political, and social environment which promotes or fights some typical way of thinking and acting: for instance, in mathematics abstraction has been viewed as a royal route of invention in Hilbert's and E. Nœther's school and, at nearly the same time, as a degenerate trend destroying the vitality of intuition in the ideology of the "Deutsche Mathematik" championed by Ludwig Bieberbach and Oswald Teichmüller. For my part, I see no unbridgeable gap between abstraction and intuition, since insights may bring in abstraction processing and follow from it as well. As some mathematicians (E. Artin, A. Weil, and others) maintain, there is indeed a symbolic and abstract intuition. Anyway, I am not aiming at discussing here the question of axiomatic or logic versus intuition, which was the focus of intense debates

3. See [Burgess & Rosen 1997, 20]. The now standard expression "way of negation" was coined by Lewis in his book [Lewis 1986].

during the 20^{th} century and still is one of the main issues of the philosophy of mathematics.

To the question "What is meaning?", I will give Quine's answer: "Meaning is what essence becomes when it is divorced from the object of reference and wedded to the word." Let me quote the whole passage:

> The Aristotelian notion of essence was the forerunner, no doubt, of the modern notion of intension or meaning... Things had essences for Aristotle, but only linguistic forms have meanings. Meaning is what essence becomes when it is divorced from the object of reference and wedded to the word. [Quine 1951, 22], see also [Quine 1990, 88]

Actually, I am taking the divorce of meaning from object of reference as a methodological device for avoiding ontological considerations and focussing on what and how we *know* rather than on what we *believe* or what we *assume* or must assume in order to give a philosophical account of some mathematical actions or attitudes. I am not saying that epistemological views do not commit to ontological assumptions, I am just saying that I will leave aside those possible commitments and the influence that they might have on actual knowledge processes and on our theoretical explanation of those processes.

1.1 Abstraction and concept-formation

Abstraction is an essential knowledge process, the process (or, to some, the alleged process) by which we form concepts. It consists in recognizing one or several common features or attributes (properties, predicates) in individuals, and on that basis stating a concept *subsuming* those common features or attributes. Concept is an idea, associated with a word expressing a property or a collection of properties inferred or derived from different samples. Subsumption is the logical technique to get generality from particulars.

This rough description complies with Aristotle's account of *ἀφαίρεσις*: Considering different things we *subtract*, remove, take away their particularities and retain only what they have in common. The concept of man applies to all humans, male or female, tall or short, blond or brown, etc.; the concept of triangle applies to *any* triangle, rectangle, equilateral or isosceles. According to Aristotle, concepts are immaterial *ideas* attached to material things; they exist *within things* on which they are predicated.[4]

There is a discussion about the nature of Aristotelian abstraction. Frege, and some Aristotle's experts such as David Ross and H.G. Apostle give a psychological interpretation. By contrast John Cleary claims[5] that *ἀφαίρεσις*

4. A little more on Aristotle's abstraction in [Szczeciniarz 1999, 4–5]. More in [Cleary 1985, 13–45].

5. See [Cleary 1985, 35–36]. On Aristotle's view about abstract objects as a result of subtraction: *τὰ ἐξ ἀφαιρέσεως λεγόμενα, τὰ δι' ἀφαιρέσεως, τὰ ἐν ἀφαιρέσει λεγόμενα* see *Metaphysics*, μ, 1–3.

that he rightly translates by "subtraction", "deprivation" (in contrast with πρόσθεσις, to which corresponds "addition"), is the logical method which is used to identify and isolate the primary subject of predication for any given attributes *(Posterior Analytics)*, and which consequently *legitimates* the intellectual separation of abstract objects.

Anyway, abstraction is the process of passing from things to ideas, properties and relations, to properties of relations and relations of properties, to properties of relations between properties, etc. Being a fundamental thinking process, abstraction has two faces: a logical face and evidently a psychological aspect that is the target of cognitive sciences.

John Locke (1632-1704) introduced *particular ideas* between individuals and *general ideas*. On a first step, particular ideas gather individuals into a class; on a second step, general ideas are *created* through the process of abstracting, drawing away, or removing the uncommon characteristics from several particular ideas. For example, the abstract general idea or concept that is designated by the word "red" is that characteristic which is common to the particular ideas (particular concepts) of apples, cherries, and blood. Thus, is pointed out the fact that the abstracting process forms a *scale* with at least two steps, and general concepts *come loose from things*. Locke writes indeed:

> General and universal belong not to the real existence of things; but are Inventions and creatures of the understanding, made by it for its own use, and concern only signs, whether words or ideas. [Locke 1689]

In contrast with Aristotle's ontological and logical point of view, Locke's standpoint is squarely epistemological. Note also that ideas may play the role of signs; later on, Charles Sanders Peirce (1839-1914) developed his semiotic philosophy on a very similar perspective.

1.2 Concepts

Developing further on Locke's approach, let us abandon Plato's and Aristotle's view that concepts are universal, unchanging *ideal objects* grasped by the understanding or made up in conformance with pre-existent relations in the real world. Concepts result indeed from the logical operation of subtraction but they do not have an eternal existence in some heavens of universal forms, separate from particulars as thought Plato or not separate as argued Aristotle. In my opinion, concepts are *historical products* of the mind's activity and their emergence depends on many theoretical, cultural, social, economical, and political data. Nevertheless concepts are or may be objective, since they help to grasp, to express in a most communicable way (at least in principle), and to master, within variable limits, phenomena of the real world.

To stress the *objectivity* of scientific concepts, the semantic tradition in philosophy [Bolzano (1781-1848), Frege (1848-1925), Husserl (1859-1938), and their followers] proposed to consider the sphere of concepts as *autonomous*. The aim was to separate semantic phenomena from their linguistic expressions and from their mental representations [*Vorstellungen*]. But grounding the semantic sphere on itself may lead to erase the historical character of its elements and to give them an immutable ontological status. It is well known that, in order to ground the objectivity of scientific concepts, Gottlob Frege proposed to locate concepts in a "third realm", the realm of "abstract objects", which are neither sensible nor mental. Frege's "abstract objects" are not *objects*, they are *meanings*, more precisely timeless everlasting meanings. Given a linguistic expression F, Frege named the meaning of F its "conceptual content" [*begrifflicher Inhalt*]. A conceptual content is either always true or always false. Frege argues that we cannot create meanings, and that we can only grasp them; he considers also meanings as if they were *a priori* essences that we have to discover. "Abstract objects" are, in Frege's perspective, "meanings in themselves", just like the old "things in themselves". That gave birth to philosophical endless and currently ongoing discussions, with a revival of Platonic tendencies.

The semantic tradition was a reaction against the promotion of the Subject by Descartes, Kant, and Hegel among others, and an attempt to "save" the alleged eternal character of scientific truths. But from a more pragmatic point of view there is no need to ground semantic objectivity on objects fixed and independent from the mind, whose accessibility would then be questionable, as pointed out P. Benacerraf [Benacerraf 1973]. The divorce of conceptual objectivity from fix and everlasting objects is not new. Georg Kreisel has pointed it out many times in his papers. Grounding on that I want to consider objectivity as resulting from a successful interaction between the rational activity of the understanding and the environment.

Concepts are also products and tools of thinking and reasoning; and they do not exist in the mind before the abstracting act. In Kant's terms they are *a posteriori*, i.e., they arise out of experience, "experience" being taken by me in an as wide as possible sense, and not in its Kantian sense, which is limited to perceptual or physical experience. I would then say that a concept is a *thought-object*, which results from a subtractive process that constructs the unity under which several specific *thought-objects*, rather than several rough physical objects, may be gathered.

Mathematical activity is concerned with thought-objects rather than with objects, even in the case of important impulse given by physical, biological, economic, or sociological phenomena. Mathematical entities are products of the activity of the understanding; they appear in a particular presentation [*Darstellung*][6], which might be modified or replaced by another one. In other

6. It is necessary to make a distinction between the word "presentation" [*Darstellung*], which means the objective mathematical way of introducing or using a concept, and the word: "representation" [*Vorstellung*], which has here its usual meaning of a subjective mental content. Moreover, when an element a of a set E

words, we have access to mathematical entities only through the concepts we form for expressing some of the properties we want to take as a basis for developing our knowledge concerning those entities and many others that come to be related to them.

> Abstraction involves perceiving something, relating it to other things, grasping some common trait of those things, and conceiving of the common trait as to it can be related not only to those things but also to other similar things. [Locke 1689, 1, 20]

A mathematical concept is the association of a meaning (conceptual content) with a sign. Generally, once adopted by a mathematical community, a sign does not change, for instance, the notation dx, the notation $\int$, or the Arabic numerals. But the meaning associated with a sign may evolve. Notably, for instance, the concept of function and the sign $\int$ have now a meaning different from the one that they had first in the 17^{th} and 18^{th} centuries; and they have now different meanings in set theory and in category theory. Actually, mathematical activity is concerned with the processes of continuous transformation of a given presentation into others: meaning changes, affording new concepts for the presumed same entity; new procedures are introduced at some point of time and reveal new aspects of our most familiar tools, new notations are proposed for designating the innovative concepts. Finally, a mathematical entity is the pair constituted by the idea of a supposed unique substrate designated by a name and its many actual and potential aspects or presentations, including the operations and rules of calculation set up in each case.[7] In other words, a mathematical entity is the virtual referent, supposedly common to similar but distinct concepts. Dedekind-Peano concept of positive integers is not the same as Euclid's concept, even though both refer to the more or less *same* entity.

Concepts are formed gradually, through reason's indefatigable abstracting work, organizing similarities and differences, dissolving hidden links and creating links that were unnoticed. They are not obvious to whom who is not trained in this kind of work. Not everybody knows Dedekind-Peano definition or even Euclid's definition of numbers. Experience rather than pure intuition is at work. New insights are gained thanks to growing knowledge and experience.

1.3 Abstract and concrete concepts

One distinguishes sometimes abstract concepts from concrete concepts. Since any concept results from an abstracting process, what is a concrete concept?

---

belongs to some equivalence class $A \subset E$, we say that a is a "representative" for A; that means that a stands for *any* element belonging to A, what again has nothing to do with a subjective (mental) representation.

7. In my view, it is hard to isolate completely the substrate entity from the operations attached to it. From an abstract point of view operations and properties are even more important than their specific substrate.

In Latin, "*concretus*" means "mixed", "composite", "compound", while the Latin word "*abstractus*" means "withdrawn", "taken out of", "extracted" (or "isolated"), "estranged". That is all that is contained in the original etymological meaning of these words. The rest pertains to the philosophical conception that is expressed through them.

1.3.1. In one sense, a concrete concept is the concept of one or many concrete sensible things: so the concepts of this table, of one apple, of five pencils, as concepts of perceived things. "Concrete" pertains to the direct *sensory* referents understood under a concept, while "abstract" hints to non-sensory referents, which are the result of a repeated operation of extracting a general idea from more particular ideas. In this sense one usually makes a radical but rough difference between concrete and abstract, actual and unreal, perceptible and imperceptible. However in science to be given to the senses is an unsatisfying criterion for the demarcation between concrete and abstract entities: elementary particles are non-sensible entities and concrete data of physical experiment. The problem of finding a criterion satisfying in any case is a difficult one, and I will not undertake to solve it because, from an epistemological point of view, the distinction between "abstract" and "concrete" is relative and unstable: a concept F may be *more* abstract than a concept G, which may itself be abstract but *less* abstract, i.e., *more* concrete than F. Leibniz said in *Nouveaux Essais* that concreteness and abstractness are correlated; that means that concreteness and abstractness are a question of more or less rather than a question of yes or no. Cognitive scientists confirm experimentally indeed the gradation of the process beginning with a direct "categorization" on perceptual objects and continuing with categorizations at higher and higher levels on more and more abstract objects.

Moreover, an interesting view comes from results of psychological experiment: concreteness is mostly associated with perceptual features of some specific situation, which is generally caught in a *global* view, while abstractness points to a wide range of diverse situations embedding different (kinds of) entities, connected in some way, and a variety of processes attached to these (kinds of) entities. And it is suggested that there is a greater engagement of the *verbal* brain (left cerebral hemisphere) system for processing of abstract concepts and a greater engagement of the *perceptual* brain system (right cerebral hemisphere) for processing of concrete concepts. An abstract concept is understood through verbal-thinking working out, a concrete concept is visualized: I have either a direct perception or at least a mental image of a table or of five apples. That may explain how mathematical working consists partly in making easier the access to mathematical concepts and their handling through visualization on the blackboard or on a sheet of paper or in the imagination: we use symbols, we draw figures and diagrams, and we write down calculations and formulae. We may even maintain that reasoning and proving through mere analysis of symbolic formulae, as in Sturm-Liouville theory of differential equations,[8] or through diagrams, as in category theory,[9] are concrete handling with abstract constructions. We manipulate formulae and diagrams as being themselves mathematical objects, detecting properties not being otherwise discerned. It is known that prodigy people who are capable to make quickly calculations with great numbers "perceive" sounds or pictures emotionally associated with numbers. Daniel Tammet says that when he is calculating the decimals of π he "sees the numerals passing before his eyes like the pictures of a movie".[10] It seems indeed that gifted mathematicians "see" the world through mathematical filters. The French neuroscientist Stanislas Dehaene thinks that

> Presumably, one can become a mathematical genius only if one has an outstanding capacity for forming vivid mental representations of abstract mathematical concepts—mental images that soon turn into an illusion, eclipsing the human origins of mathematical objects and endowing them with the semblance of an independent existence. [Dehaene 2011, 225]

The irresistible leaning to a realist view of the mathematical universe of concepts and techniques has its roots in the actual process of visualizing abstract procedures.

1.3.2. In a second sense, "concrete" pertains to our usage and training. Familiar concepts are taken to be concrete and intuitively (visually) graspable, e.g., the positive integers, which are called "natural numbers" qua being the basic representation of the act of counting. Thus concreteness is a developed or developing character. According to Kant's *Logic*:

> The expressions *abstract and concrete* refer not so much to the concepts themselves—for any concept is an abstract concept—as to their *usage*. And this usage can again have different grades;—according as one treats a concept now more, now less abstract or concrete, that is, takes away from or adds to it now more, now fewer definitions. [Kant 1800, § 16, Anmerk 1, 154]

In this second perspective, the distinction abstract/concrete is clearly an *epistemic* distinction and it is *relative* in a sense different from that meant by Leibniz: not only abstract and concrete are correlated concepts, but an abstract concept or construction may become concrete or more concrete and it may be visualized through a symbol or image or diagram standing materially for it. That means in fact that we may form, through some kind of drawing, concrete representations of abstract concepts.

8. Poincaré called that "qualitative analysis" [Poincaré 1928, XXI–XXII].
9. See [Krömer 2007, especially, 81–84]: *commutative* diagrams play a central role.
10. [Tammet 2005]. When D. Tammet multiplies two numbers, he "see[s] two shapes. The image starts to change and evolve, and a third shape emerges. That's the answer. It's mental imagery. It's like maths without having to think".

Hegel (1770-1831) introduced important refinements in the distinction concrete/abstract. He assumed that any concept is always abstract, but he added that *a genuine concept* is not only abstract, but also concrete, in the sense that its definitions (what old logic calls features) are combined in it in a single complex expressing its individual unity. A concept is concrete because it contains all the content of its genesis within it. By contrast immediate perception is abstract in the sense that its determinations remain undeveloped.[11] A concept is the concrete unity of different determinations. Thus the concreteness of a concept lies in the *meaningful cohesion* of its features, which may be developed at different moments of time. For instance, out of context, a verbal definition is abstract and abstract only. Immersed into the context of a scientific theoretical discourse, any abstract definition becomes concrete (in an epistemic sense). The concreteness of a concept is therefore always expressed through unfolding all its possible definitions/features in their *mutual connections* rather than through an isolated "definition", and in immersing the concept into a web of interconnected concepts. It is as to say that "flesh" is given by the mutual connections between different features of the concept under consideration and by the links with other concepts. Such a consideration may well be applied to mathematics: the image of a dense network for representing the mathematical stuff has become commonplace by now.

2 Mathematical practice

2.1. Mathematical concepts may generally be introduced or defined in different ways. The more presentations [*Darstellungen*] a concept has and the more it is embodied by different procedures performed in different areas, the more concrete it is taken to be. This may happen through two ways.

a) When a concept is repeatedly used in different contextual theories, e.g., when we *add* numbers, vectors, vector spaces, etc., we get a *meaning-generality* (semantic generality), which is an *extensive generality*, a transversal generality of use. In the same wise, we use "products" for vector spaces, groups, topological spaces, Banach spaces, automata, etc. In each case we have to tell which properties among all the possible properties of the operation $+$ or $\times$, such as commutativity, associativity, etc., are preserved and which must be dropped. The fewer are the properties considered, the greater is generality. The very general concept of addition is illustrated by the structure of a monoid, which is instantiated by so many different models. The concreteness comes from the repeated use under rules specified in each case.

11. I am simplifying the very suggestive although intricate developments of Hegel's *drittes Buch* of [Hegel 1812-1816].

b) The more connections a concept has, actually or potentially, with other concepts, the more intricate is its own meaning. Through its *meaning-complexity* (semantic complexity or richness)[12] a concept gets some kind of concreteness, it looks like an individuated entity because several operations along with their properties are combined under it. Concreteness here is taken in Hegel's sense. In contrast with the traditional ratio between extension and intension of a concept, it is not the case in mathematics that increasing meaning complexity entails decreasing meaning generality; for instance, a category, let us say *Grp* is at the same time a more general and a more complex mathematical object than the group structure. Thus, analysis of mathematical abstraction does not give the same results as the traditional grammatical or logical analysis of concept formation.

2.2. Creative manipulation of mathematical concepts pertains their meaning, not just their names or nominal definitions. Names designate things, while concepts condense meaning even when they appear at first sight very abstract.

For instance real numbers may seem so abstract that their *mathematical existence* is challenged. They are, indeed, rejected by some constructivist mathematicians: e.g., instead of speaking of *real* roots of an algebraic equation, Kronecker considered intervals bounded by *rational quantities*, rational quantities being *constructed* by a finite number of operations from the integers. However, there is a larger notion of constructive existence, as it was made explicit by Hermann Weyl, who argued that we are entitled to claim that there exists an α only after having *instantiated* α [Weyl 1921, 54–55].

In this view, real numbers *exist* since we have encountered instances of them (e.g., ratio of the side of a square to its diagonal, π, the base e of the natural logarithm). The *concept* of real number, though abstract in the double sense that we can neither survey *all* its individual instances nor have a finite calculation for each instance, needs not to be eliminated; we rightly reason with *the concept of real number* as a *set*, a collection, and as a domain equipped with more than only one *structure*. Putting a structure on a set is stipulating relations and operations (functions) between the elements of the set and stipulating rules for working with them. In addition to algebraic structures such as groups, rings, fields, modules, vector spaces, etc., we have order structures, metric structures, topologies, differential structures, categories, among others.

The *structural* complexity of the real number system emerged gradually (and mainly in the 19^{th} century) through successive abstractive operations,

12. I thank one of the referees who suggested to use "richness" rather than "complexity". I am taking indeed meaning-complexity not as a kind of mathematical complexity, algorithmic or measurable in some other way, but as an expression for the *polysemous* character of many, if not all mathematical concepts. "Number", for instance, has different meanings depending on whether you consider integers or rational numbers, or real numbers, or quaternions, etc. The polysemous character of mathematical concepts and symbols has been put to the fore by the rise of abstract axiomatics (comments might be found in [Benis Sinaceur 1991]).

disentangling different structures that were mixed together, dissociating especially topological notions from algebraic operations, from the order relation, and from the metric. As a bearer of several structures, the real numbers appear *compound* and multi-faceted, just like individuated physical objects.

Actually, the set of real numbers carries the following standard structures:

- an order: each number is either less or more than every other number.
- an algebraic structure: multiplication and addition make it into a field.
- a measure: intervals along the real line have a specific length, which can be extended to the Lebesgue measure on many of its subsets.
- a metric: there is a notion of distance between points.
- a geometry: it is equipped with a metric and is flat.
- a topology: there is a notion of open sets.

More significant is the possibility of hybrid structures; for instance:

- the order and, independently, the metric structure induce the standard topology,
- the order and the algebraic structure make this set into a totally ordered field,
- the algebraic structure and the topology make it into a Lie group.

What matters with a structure, that was called "concept" by German mathematicians of the Göttingen School, is that it provides us with a new abstract concept, and, at the same time, it gives a more determined meaning to the underlying set of specified or unspecified elements. It is to say that abstraction brings a richer, not a poorer meaning, even for more general concepts. In other words, structural complexity brings simultaneously syntactic and semantic richness. As W.v.O. Quine stressed many times, the creation of abstract concepts is a semantic ascent,[13] which goes hand in hand with the syntactic ascent.

2.3. Thus, we observe in mathematics something which is close to Hegel's description of abstract and concrete. What is of concern to us in this description is that it develops further Kant's *epistemological* distinction between abstract concept and concrete concept.

According to Kant, very abstract concepts give little information about many things, while through concrete concepts we know much about few things.

13. The semantic ascent "is the shift from talk of miles to talk of "mile", it is what leads from the material *(inhaltlich)* mode into the formal mode, to invoke an old terminology of Carnap; [...] The strategy is one of ascending to a common part of two fundamentally disparate conceptual schemes", [Quine 1960, 271–272]; [Quine 1990, 33].

To account for the fruitfulness of mathematics, Kant argues that mathematical knowledge proceeds *in concreto*, i.e., presents the concept into a pure and *a priori* but *singular* intuition. Thus the division abstract/concrete integrates the division general/particular and the division class/individual. Kant tells that the concrete usage of a concept is that which is most close to the individual.

By contrast, Hegel considers not only the form of knowledge, but also its content, and he detaches the concreteness from its reference to a real-world existent individual: we may have a very abstract, a very poor knowledge of an individual or of a singular situation, while a concept, as a product of knowledge, is an *evolving concrete unity*, which may get more and more meaning determinations and, then, become more and more concrete. We thus go *from abstract to concrete* and not vice versa. What matters is how much and *via* how many ways or viewpoints we *know* about something at some point of time; what matters is the *knowledge-content*, the increasing richness and the progressive diversification of knowledge. *Knowledge-content is semantic content in its historical dimension.* Abstractness and concreteness are not fixed forms of the subjective, empirical or transcendental, act of knowing, they are characteristics of *knowledge as such*, of knowledge as *historical and objective* product of collective activity. An important gain of that view is that it is now clear that the division abstract/concrete coincides *neither* with the division general/particular *nor* with the division class/individual.

2.4. Let us return to mathematical practice. If an individual thing (phenomenon, fact, entity, concept, procedure, theory) is not understood through the concrete interconnection within which it actually emerged, exists, and develops, that means that only *abstract knowledge* has been obtained, e.g., when one has learned what is a group by learning the group axioms without knowing the context of their emergence (history) and at least some of the different situations where they can be applied fruitfully for revealing the structure of a domain or suggesting a solution for some problem (actual practice and problem-solving). Usually, algebraic concepts produce knowledge when they are tied to facts and problems belonging to other mathematical areas, arithmetic, geometry, analysis, topology, etc., or belonging to an earlier stage of the algebraic trend itself, as it is, e.g., the case for the concept of group. If, on the other hand, an individual thing is understood in its objective links with other things forming a coherent network, that means that it has been understood, realized, known, conceived *concretely*. In such a perspective we can understand how we may have a *concrete knowledge* of a highly abstract concept, as it happens especially in modern mathematics. Thus an abstract concept becomes concrete not only through its instantiations (realizations, models), but also through the theories in which it plays a role, i.e., through the theoretical or technological applications following from it, and still through the theories to which it gives birth by being included in a more general abstract concept.

For instance, the algebraic concept of group is made concrete 1) through its embodiment in arithmetical and geometrical models, 2) through its use to represent symmetry in physics and to classify crystal structures in chemistry, and also 3) through the categorical construction of *Grp*.

3 Descriptive analysis of the fundamental thinking processes underlying the main ways of getting abstraction

The title of this section seems ambitious. However, I must say that since I am no expert in cognitive sciences, I am essentially relying on a more or less direct analysis of actual mathematical procedures combined with information got in cognitive scientists' readings. Cognitive scientists name "categorization" any kind of activity that involves association, comparison, analogy, and correspondence between two or more things. I will detail the actions performed in such an activity, which is in fact the task of getting abstract ideas, from the most simple to the most sophisticated.

3.1. Abstracting is a result of several *overlapping* or *intertwined* thought operations that I describe now.

- Considering things, not necessarily physical ones, not necessarily located in space and time.
- Comparing things *not in themselves* but *sub specie generalitatis*, i.e., comparing them as possible samples of something else, something which is not necessarily already known but only glimpsed and still relatively vague or fuzzy. Precision comes later.
- Selecting one or several aspects (qualities, properties, predicates) in the things submitted to comparison and presumed to have something in common, then presumed to be *classed* (subsumed) under some *concept*.
- Leaving aside or discarding all other aspects, especially specific substantial or space-time aspects. This operation has been called idealization because it comes down to extracting a *form* from sundry situations; it has been especially promoted in the beginning of the 20^{th} century by abstract algebra and abstract topology, which made familiar the study of structures not qua being associated with any specific instance. Idealization follows from seeing or guessing some *invariant* basic properties attached to a plurality of apparently heterogeneous situations and it leads to a unifying view of the different domains on which we perform the same type of operations: counting, addition, subtraction, compactification, etc. Idealization has also a *heuristic* role in suggesting a possible

or unexpected connection with a situation not having been considered at first. The extracted form is not rigid; it may be affected by some controlled variation in passing from a certain type of situations to another one: the addition of two vector subspaces differs from the addition of let us say two real numbers.

– Isolating some property or some set of properties of the operation(s) under consideration and viewing them *on their own*, i.e., transforming the selected conjunction of predicates into a thought-object (Frege's radical separation between concept and object does not fit mathematical practice). Peirce called this kind of transformation "reflective" or "hypostatic" abstraction, Husserl called it "thematization". Cavaillès popularized the term "thematization", at least among French philosophers.

  Thematization is especially important in considering as a whole an *infinite* collection of things; it played a fundamental role in the emergence of set theory, and it has been consistently codified within different frames: Russell type theory, Zermelo-Fraenkel system (ZF) and Quine's system (NF).

  Thematization is essential in passing from a set S of elements to its possibly many structures and from the study of a structure Σ on the set S to the study of the structure Σ in its own right, i.e., to the study of a class of homomorphisms between structures of the type Σ. The standard example is given by the passage from Dedekind's axiomatics for numbers and Hilbert's axiomatics for geometry to Emmy Nœther's style of studying classes of group's homomorphisms, classes of ring's homomorphisms, etc. Attention is paid to homomorphisms rather than to the sets that are respectively source and target of them. It is that attitude that "changed the face of algebra" (see [Artin 1962, 555] and [Weyl 1935, 433]) opening up a wide domain of research and new stuff for developing new insights and new procedures typical of the "*begriffliche Mathematik*",[14] which was understood as the study of algebraic or topological structures considered in and for themselves.

  Thematization plays also a role in transforming an abstract object (predicate, concept matching many items possessing similar structures) into a concrete object, which becomes *element* of some larger class, e.g., the structure of abelian groups viewed as an element of the category of groups.

  Thematization is still involved in analyzing a concept by breaking down its global unity into components that were formerly tightly connected. Analysis, in this chemical sense, comes out at idealization and thematization; it is disambiguation of meaning by dissociating and studying separately characters, which have been "intuitively" associated during centuries. It was, e.g., the case when Riemann showed (1854) that not

14. The expression "*begriffliche Mathematik*" was coined by Pavel Alexandroff in his obituary of Emmy Nœther [Alexandroff 1935].

every space is a metric space or when Dedekind (1872 or even sooner) showed that not every space is a continuous space. Thus the concept of space becomes very general, divested from any particular property, and simultaneously subject to different specifications. By space we understand now any set of elements taken as a substrate for some selected relations and functions and as specifications we get new *subclasses* of objects, in our examples the subclass of metrical spaces and the subclass of continuous spaces.

More generally, by iterated thematization one pushes further mathematical conceptual constructions, as it is well illustrated by category theory, which is a theory of systems of structural theories, treating the notion of structure in a uniform manner: e.g., sets and usual functions form the category of sets (Set), groups with group-homomorphisms (which preserve the group-structure) form the category of groups (Grp), topological spaces and continuous functions (which preserve the topological structure) form the category of topological spaces (Top).

Abstracting again, functors are structure-preserving maps between categories. Functors (arrows) are the very objects of category theory; they belong to a higher level of abstraction than morphisms, which in their turn are on a higher level of abstraction than maps. By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures.* This is a fundamental idea, which first surfaced in algebraic topology. Searching for general invariants makes up the dynamic construction of new layers of sophisticated abstraction processes. Abstracting yet again, a "natural transformation" provides a way of transforming one functor into another while respecting the internal structure (i.e., the composition of morphisms) of the categories involved.[15] Hence, natural transformations

15. If F and G are functors between the categories C and D, then a natural transformation η from F to G associates to every object X in C a morphism $\eta_X : F(X) \to G(X)$ between objects of D, called the component of η at X, such that for every morphism $f : X \to Y$ in C we have:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X$$

This equation can conveniently be expressed by the commutative diagram:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

The notion of a natural transformation states that a particular map between functors can be done consistently over an entire category. Informally, a particular map, let us say an isomorphism between individual objects (not entire categories) is referred to as a "natural isomorphism", meaning implicitly that it is actually defined on the entire category, and defines a natural transformation of functors; formalizing this intuition was a motivating factor in the development of category theory.

can be considered to be "morphisms of functors"; they yield the usual homomorphisms of structures in the traditional set theoretical framework. And so on...

The point is the endless *dynamical* concatenation of polysemous symbols and symbolic operations. Of course, this concatenation is not necessarily linear; it forms a kind of tree with interweaved branches at the same level and from lower to higher levels, or, as said above, a complicate and dense network.

– Analogies are concurrent with idealization and thematization. Setting up, guessing, or looking for analogy[16] between sundry situations is a main way to bring to light similarities, differences and possible relations between two or several thought-objects. Combined with idealization and thematization, analogy is a basic constituent of abstraction.

One makes sometimes a distinction between analogy and abstraction. Grounding on the emergence of abstract group theory from 1) the theory of algebraic equations, 2) number theory, and 3) geometry, and on the conception of modern algebra as the study of algebraic structures which came *after* 1) abstract group theory, 2) abstract field theory and 3) abstract ring theory, Jean-Pierre Marquis argues that it is an empirical fact that analogy concerns two things, while abstraction comes only when three or more things are considered [Marquis forthcoming, 5–6]. Reasoning by analogy is indeed transferring information or meaning from a particular situation to another particular situation. A good example is given by J.-P. Marquis, namely Dedekind's and Weber's work on algebraic number theory and algebraic functions. Another example is the transfer of algebraic laws and tools to logic in the works of G. Boole, A. de Morgan, E. Schröder, etc. Abstraction comes in play when several, and not only two, domains of entities or several classes of structures are *a priori* in question.

Indeed, at a first step a theory is abstract when it has *a priori a plurality of models*. The plurality criterion is indeed commonly used to distinguish between concrete or material axiomatics and abstract axiomatics, e.g., between Euclid's geometry and Hilbert's axiomatization of Cartesian geometry,[17] which permits to construct different geometric models by selecting different sets of axioms. At a second step, domains of entities are neglected, while one considers *a priori* a plurality of structures along with their specific structure preserving morphisms.

But, even in the earlier stage of considering similarities, differences and relations between only two situations belonging to the same domain (or only two domains of different entities or only two structure types) is

16. I have analyzed different aspects of the fundamental role of analogy in the progress of mathematics in [Benis Sinaceur 2000].

17. [Hilbert & Bernays 1934-1939, 20]. In Hilbert's and Bernays' terms the distinction is between "*inhaltliche und anschauliche Axiomatik*" and "*formale Axiomatik*".

involved the implicit assumption that it must be some abstract framework in virtue of which the transfer from one situation to the other or from one domain to the other or from one structure type to the other is possible. Analysing how analogy works, Henri Poincaré writes that the mathematician must have a direct insight of what makes the organic unity of sundry situations.[18] Analogy as a guide for mathematical invention and for great productivity with economy of thought is the chief theme of Poincaré's talk at the 1908 International Mathematical Congress. According to Poincaré, the crucial step is the passage from material to formal[19] and from diversity to unification: analogy between materially different entities or procedures appears when one sees, constructs, or supposes a formal similarity between those entities or procedures.[20] Formal similarity hints to a unique mould, which may serve for predicting or finding out unexpected analogies with new items and which may thus lead to a more precise view of the architecture of the whole body of mathematics, as it happened with the concept of group. Thus, searching after analogies involves an abstracting mind, if not yet a systematic use of the abstract method.[21] There is actually a back-and-forth play between analogy and abstraction: setting up analogies leads to conceive of an abstract theory and, once an abstract theory is at hand, it is used to unearth more and deeper analogies.[22]

18. [Poincaré 1900, 127–128]. Notice that Poincaré used "unity" rather than "identity".

19. Poincaré is using the term "formal" as the opposite of "material" and he underlines the important role of language in discovering new analogies between domains sundry at first sight, but he does not mean a logically formal language, as it is meant in Hilbert's and Bernays' *Grundlagen*.

20. "En mathématiques, [...] des éléments variés dont nous disposons, nous pouvons faire sortir des millions de combinaisons différentes; mais une de ces combinaisons, tant qu'elle est isolée, est absolument dépourvue de valeur ; [...] Il en sera tout autrement le jour où cette combinaison prendra place dans une classe de combinaisons analogues et où nous aurons remarqué cette analogie; nous ne serons plus en présence d'un fait, mais d'une loi. Et, ce jour-là, le véritable inventeur, ce ne sera pas l'ouvrier qui aura patiemment édifié quelques-unes de ces combinaisons, ce sera celui qui aura mis en évidence leur parenté. [...] Si un résultat nouveau a du prix, c'est quand en reliant des éléments connus depuis longtemps, mais jusque-là épars et paraissant étrangers les uns aux autres, il introduit subitement l'ordre là où régnait l'apparence du désordre. [...] ce n'est pas seulement l'ordre, c'est l'ordre inattendu qui vaut quelque chose" [Poincaré 1908, 168–170].

21. See J.-P. Marquis' fine decomposition of the abstract method into four components in [Marquis 2012, 9–10].

22. Saying analogy or similarity is not saying identity. While mathematicians are using analogies to set up isomorphisms between sets or equivalence between categories, some cognitive scientists are using the mathematical concept of isomorphism for giving a theoretical explanation of analogy (see e.g., [Gentner 1983, 155–170]).

3.2. Although I have taken examples mainly from modern mathematics, it must be stressed that abstraction is there from the very first beginning. Even the most elementary notions of mathematics are abstract: the notions of number, of rectangle or triangle or circle, etc., are abstract notions, i.e., products of abstracting processes. For instance, whole positive numbers result from several abstracting processes: associating a symbol with a collection of actual things, dissociating this symbol from this particular collection and associating it with any collection of the same number of things, then establishing a one-to-one correspondence between many different collections, combining this symbol with other symbols similarly generated in order to perform operations like addition, multiplication, and so on. It is only through a long habit that we consider positive integers as given intuitive *concrete* objects and geometrical figures as concrete spatial visualizations supporting the proof process. Abstraction is always there and is an ongoing process, becoming more and more sophisticated. As Ch. S. Peirce, E. Husserl and J. Cavaillès argued, abstraction is "constitutive of" mathematical thinking and it can be repeatedly exemplified in the processes of idealizing, thematizing, extracting invariants, and setting up analogies. The more advanced the abstraction process, the more concrete the abstract objects become—classes, structures, operations as such, functions as such, morphisms, categories, etc. Thus it is not a paradox to think that, in mathematics, higher levels of abstraction produce more and more concrete thought-objects, concrete in the double sense that they are complex, individuated objects with various determinations, and that they become concretely known and manipulated through symbolic formulation, precise diagrams or even sketchy drawings. "Concrete" means simultaneously polysemous[23] and daily handled.

3.3. Ascent towards abstraction is not limited to the logical process of *subsuming* particulars or particular ideas under a more general concept. Notably Frege rejected the Aristotelian ἀφαίρεσις as being *not* the only sort of logical abstraction[24] and he dissented from the traditional view on concepts; he used mathematical tools, namely a functional relation and an equation for stating a putative logical definition of the concept of number. In most elementary cases indeed a mathematical concept encompasses more thought-processes than only the logical subsumption, to which corresponds the set-theoretic operation of inclusion. In practice mathematicians are dealing with many sorts of operations and calculations and many sorts of relationships between structured sets (one-to-one correspondence and equivalence relation as in the so called Hume's principle,[25] linear transformations, group homomorphisms, morphisms, etc.).

23. On the polysemy or ambiguity of axiomatic concepts, see e.g., [Benis Sinaceur 1991, 191–196].

24. Actually, Frege thought that Aristotle's analysis was psychological.

25. The name "Hume's principle" was coined by George Boolos. This principle plays a central role in Frege's definition of numbers, and it says that the number of

Subsumption is a fundamental level of classification, but in mathematics fruitfulness and new insights result from combining it with other abstraction processes, as Frege made clear in his seminal reform of logic and as I shall illustrate below by some mathematical examples. I am not saying that the ascent towards abstraction is not a logical ascent from step to step. I am just saying that, in mathematical practice, at any step, genuine mathematical stuff fills the logical move. This is why I have stressed hereinabove that the ascent is at once semantic and syntactic. Mathematical abstraction is a many-faceted and multi-leveled process and it leads to a sophisticated and branched hierarchy of mathematical concepts and operations. Moreover it is not always the case that the more abstract a concept is the more undetermined it is. For instance, with just a general concept of set as a collection of any things one does not go far. If one wants actual and effective work, one must begin by a meaning determination, i.e., by setting up the axioms ruling a consistent usage of the concept. It happens often that the more abstract is a structure the more overdetermined and stratified it is: axiomatics and category theory give many examples. The mathematical branching of concepts is simultaneously complication of concepts taken in isolation and clarification of their mutual links: bringing to light new and new structures gives more and more power to solve problems not one by one depending on their particularities but uniformly in one go grounding on the general structure fitting all of them.

3.1 Abstraction and axiomatization

A rapid look at the history of mathematics, especially of modern mathematics, shows that abstraction is closely tied up with symbolization and axiomatization. Mathematical thinking is thinking with and on symbols and diagrams, may they be considered as representations or as themselves mathematical objects. Anyway, creative manipulation of symbols and diagrams does not dwell only on their drawings; it pertains their meanings and meaningful connections with other symbols and diagrams. Abstract concepts (abstract structures) are usually defined by a finite set of axioms that state the relations to be satisfied by candidates for being models of those abstract concepts. But abstract concepts need not to coincide in every respect with their less abstract counterparts; the meaning changes in between,[26] it becomes more sharply determined and yet more ambiguous: not every group is abelian; the multiplication of

---

Fs is equal to the number of Gs if and only if there is a one-to-one correspondence (a bijection) between the Fs and the Gs. Boolos and other logicians as well have recognized that Hume's principle is *not* a logical truth, but from it we can logically deduce what we now call second-order arithmetic. See [Boolos 1998] or [Zalta 2013].

26. The fact is stressed by J.-P. Marquis in the case of the passage of set and mapping to abstract set and arrow. Marquis writes : "Abstract mathematics, like the concept of mathematical structure, is open, in the sense that it denotes changes with respect to the theoretical tool used to interpret and illustrate the concept" [Marquis 2012, 2, 11 sqq.].

integers is symmetric, composition of permutations of three objects is not; in category theory the term "structure" has not exactly the same meaning as it has in set theory and in model theory: structures of structures do not always reduce to structures of elements (see [Awodey 1996]). There are really different levels of abstraction, even if there are connecting paths between levels.

Mathematicians differently oriented have recognized axiomatization as an effective tool for understanding and invention: Dedekind, Hilbert, Emmy Nœther, and Emil Artin, but also Poincaré, Weyl, and Brouwer,[27] who did not reject the use of the axiomatic method but rather the view that it might provide a foundation or dispense with calculation and algorithmic proofs. One must distinguish between axiomatics as a fruitful mathematical method and axiomatics as a putative foundation or useless mathematical ideology, which is an epiphenomenon harmful in teaching. In practice, it would be absurd to go without the axiomatic contributions: for instance Galois' theory has been deeply and effectively understood only after Dedekind's, Weber's and Artin's axiomatic presentations. Working with axioms develops new insights and ideas: notably the study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures *directly* from the axioms of a category. Mathematics is always aiming at more and more general results about more and more complicated structures.

Although axiomatization plays now an indispensable role in mathematical practice, it is not the only way to make mathematical procedures abstract. I will now give a non-exhaustive list of other mathematical abstraction processes that interplay in mathematical thinking and actually illustrate the unceasing iteration of intertwining processes of setting up invariants, idealizing entities and procedures, transforming operations into objects (thematizing), bringing to light analogies between sets, structures, categories, etc.

4 Various samples of mathematical abstraction processes

1. Representing an *infinite* numerical sequence by its *law* of recurrence. One gets the law by discarding concrete *calculation* and retaining only how one passes from *any* element n to its successor. One does not actually know all the elements of the sequence but one knows how to *generate* the sequence. Here it matters of finding out a *rule* of calculation, not a concept, but the rule

27. See e.g., Poincaré's praise of the concept of group [Poincaré 1908]; Weyl, [Weyl 1932, 349] and [Weyl 1951, 464]; Brouwer's conception of geometrical method [Brouwer 1909].

dispenses with enumerating all the elements of the sequence like the concept of even integers dispenses with enumerating all the multiples of 2.

2. Discarding the specific nature of the elements forming a sequence, e.g., the sequence of positive integers, so as to characterize the *order type* of the sequence. For that Dedekind invented the concept of *chain*.[28] What matters here is neither the integers themselves nor even their generative law by itself, but the ordering generated by this law (linear discrete order). The level of abstraction is higher than in the example 1, because we are not concerned with a particular calculation law valid for one particular sequence but with a law type generating an order structure suitable for integers and for sequences of unspecified elements as well. Dedekind's definition shows that integers are a particular instantiation of a general structure; it indicates one way of linking abstraction and generalization.

3. Combining operations (+, $\times$, etc.) and selecting properties of these operations (associativity, commutativity, etc.) in order to form different kinds of mathematical structures (concepts): groups, fields, rings, ideals, lattices, vector spaces, categories, etc., that connect models originating from different mathematical areas or different structures. Abstract concepts are *multiply instantiated*, they define not one single model nor a single structure, but *classes of models and classes of structures.* This kind of abstraction is really "modern", in contrast with Euclid's axiomatic system for geometry, which concerns one single model (the real three-dimensional space) of one single structure (the structure of Euclidean space, realized for instance by the vector space $\mathbf{R}^n$ with the standard inner product and by the vector space of real polynomials of degree $\leq n$ with a convenient inner product). In the spirit of Hilbert's and Bernays' distinction one sets Euclid's material system in contrast with Hilbert's system in *Die Grundlagen der Geometrie* (1899) or Dedekind's system for arithmetic [Dedekind 1888], which are abstract systems (informally presented); moreover one makes a difference between Dedekind/Hilbert's style and Emmy Noether's style of abstraction.

In the perspective of abstract set theory we are using, for instance, the following terms, which mostly appeared in the 19^{th} century:

- "abstract set", which surfaces in Cantor's matured theory,
- "abstract group": Dedekind recognized similarities among various mathematical structures, like rotations and quaternions, and identified them as instances of the abstract notion of group [Dedekind 1855-1858, 439]. Heinrich Weber gave, in 1882, axiom systems for groups, and later on

28. A chain is the minimal closure of a set A in a set B containing A under a function f on B (where being "minimal" is conceived of in terms of the general notion of intersection).

these axiom systems have been formalized and investigated in their own right.

- "abstract number", which was used, for instance, by Bolzano in the sense of number as a single entity and in contrast with concrete number, which is number associated to the things being counted [Bolzano 1851]. We owe the abstract axiomatic characterization of the sequence of positive integers to Dedekind through the definition 73 of [Dedekind 1888]:

> If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called natural numbers or ordinal numbers or simply numbers, and the base-element 1 is called the base-number of the number-series N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. The relations or laws which are derived entirely from the conditions α, β, γ, δ in (71) and therefore are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements (compare 134), form the first object of the science of numbers or arithmetic.

- "abstract field": this structure has been defined by Steinitz [Steinitz 1910].

- "abstract space": it surfaced in Riemann's famous paper [Riemann 1854], where a topology and a metric for a space E is defined *before* defining the functions having their arguments and values in E. From 1914 onwards ([Hausdorff 1914]) it was known that a topological space was a set structured by a lattice of open subsets. But it was not until the middle thirties, with the work of Marshall Stone (1903-1989) on the topological representation of Boolean algebras and distributive lattices that this connection between topology and lattice theory began to be exploited, and it became clear that it is possible to construct *topologically interesting* spaces from purely *algebraic data*.

- In the categorical perspective we are using "morphism", which is the abstract generalization of structure-preserving mappings between two mathematical structures. In set theory, morphisms are functions; in linear algebra they are linear transformations; in group theory, they are group homomorphisms; in topology, they are continuous functions, in manifold theory they are smooth functions (functions having derivatives of all orders), and so on.

4. Classifying: This action may be direct as when one collects elements in a set—by the way it is epistemologically meaningful that for "collecting" Kronecker said "*begrifflich zusammenfassen*", expression that might have been come under Dedekind's pen, while Cantor used the presumptive ontological "*zusammensein*"—, or when one collects "material" (interpreted, embodied) structures under the head of an abstract structure of which they are models, or when one collects abstract structures in a category, or when one ranks categories under different types: abelian categories, Cartesian closed categories, complete categories, topos, etc.

A more stratified task consists of dividing a set in classes of equivalent elements[29] and making up the identity of a class from the equivalence of its members: quotient group, quotient ring, quotient field, etc. Equivalent Cauchy sequences of rational numbers are identified for defining the concept of real number. Similarly, Frege used the process of forming equivalent, namely equinumerical classes for *defining* positive cardinal numbers. Russell named this kind of definition "*the* abstraction principle"; it is the subject of many philosophical reflections, but in mathematics even though it is systematically used, it is *only one* abstraction principle, *only one way* to perform abstraction, namely forming a quotient structure of some given structure. In particular, this way must not be confused with those listed in 2. (order structure) and 3. (algebraic structure), where, considering a material structure, we do not start by defining an equivalence relation on the underlying set of elements, but we consider the schematic structure itself, independently of the material elements, and examine which compatible relations may be matched for a characterization. Such structural definitions were not welcome in Frege's conception. Frege's abstraction principle was not a mathematical novelty; the novelty lied in introducing a typical mathematical relation, the one-to-one relation, within the scope of logic and presenting this relation as a logical tool for defining a concept.

More generally, the equivalence relation is involved in "classification theorems", which answer the question: "What are the objects of a given type, up to some equivalence?" Example: the Wedderburn theorem (1908), which states that every simple ring that is finite-dimensional over a division ring (a simple algebra) is a matrix ring; it is a way to unify the real numbers, the complex numbers, the quaternions and the square matrices under the same structure. Emil Artin later (1928) generalized this result to the case of Artinian rings (rings satisfy the descending chain on ideals). Several levels of abstraction are crossed from the abstract concept of ring to Artin's theorem. Another famous example is the classification of finite simple groups: every finite simple group belongs to one of four classes (cyclic groups, alternating groups, classical Lie groups, sporadic simple groups). In category theory equivalence is very essential: one reasons on equivalent categories, i.e., categories related by a functor

29. Elements of an equivalence class satisfy a relation, which is reflexive, symmetric, and transitive.

F, which has an inverse G, but the composition of F and G is not necessarily the identity mapping; thus equivalence of categories is less restricted than isomorphism of categories and allows to translate theorems between *different kinds* of structures.

5. Classification is a down top process. Going top down, the converse action is also a way to show the structure of an entity or a procedure by breaking it up into simple pieces: e.g., reducing, factorizing a number, a polynomial, an ideal, in order to unearth the basing building blocks. Generally factorization and classification blend together for producing what was named "structure theorems" in the 1930s.[30] For instance, Kronecker proved that every finite abelian group is uniquely presented as a direct product of cyclic groups of prime power order; this theorem applies to Galois' theory, to number theory, and to other theories; it is generalized to finitely generated abelian groups and to finitely generated modules over a principal ideal domain;[31] in the latter case the structure theorem roughly states that finitely generated modules can be uniquely decomposed in much the same way that integers have a prime factorization. That shows deep connections between arithmetic and algebra: historically that was a result of the project, shared by Kronecker, Dedekind and Weber, to arithmetize algebra, i.e., to bring to light the analogy between divisibility of the integers and divisibility of ideals in a ring.

6. Thinking in terms of *functional* relation, so as to make room for establishing other identity relations than equality of elements of some set, or equinumericity between different sets, or isomorphisms between distinct models of this or that structure. In set theory one associates frequently an element a belonging to a set S to an element α belonging to a set Σ, and one reasons on α as "representative" for a. Although one may describe this process by saying that it consists in *seeing a as* an α, one must underscore that what is at stake is not the mental content of an idea [*Vorstellung*], which would consists in a *psychological* association of α with a; what is at stake is the presentation [*Darstellung*] of something as something different but similar in some respect, more exactly the *functional* association of α with a, which makes $\alpha = f(a)$. It may happen that it is much easier to get results by reasoning on the image α rather than directly on the source element a, and then to come back to a adjusting the obtained results. Dedekind saw a very fundamental way of mathematical thinking in "the ability of the mind to relate things to things, to

30. This expression was commonly used; one can find it for instance under Helmut Hasse's pen [Hasse 1931, 496] (see [Benis Sinaceur 1991, 187–191]).

31. Principal ideal domains (PID) behave somewhat like the integers, with respect to divisibility: any element of a PID has a unique decomposition into prime elements (so an analogue of the fundamental theorem of arithmetic holds); any two elements of a PID have a greatest common divisor, although it may not be possible to find it using the Euclidean algorithm.

let a thing correspond to a thing, or to represent a thing by a thing" ["...*Dinge auf Dinge zu beziehen, einem Dinge ein Ding entsprechen zu lassen, oder ein Ding durch ein Ding abzubilden"*]. Indeed, a real number x is associated with a certain class of equivalent Cauchy sequences (x_n) of rational numbers, a rational number p/q may be identified with the equivalence class of the ordered pairs of integers (p,q) with $q \neq 0$, modulo the relation (p,q) (p',q') iff $pq' = qp'$, etc.

More generally, when a structure A is embedded in another structure B by an injection f, every element a of A is identified with its image $f(a)$, in B. $f(a)$ is another way to present a, which then has a *multiple identity* or, more exactly, we have for a several distinct representatives that we identify as referring to the *same* entity. When f is a bijection, a and $f(a)$ are distinct but behave in the same way in the structure A and the structure B respectively, A and B being isomorphic. This process turned out to be essential in category theory. As spotted by J.-P. Marquis: "There is no unique, global, and universal relation of identity for abstract objects. [...] Abstract objects are of different sorts and this should mean, almost by definition, that there is no global, universal identity for sorts. Each sort X is equipped with an internal relation of identity but there is no identity relation that would apply to all sorts."[32]

In mathematics, one looks permanently for new presentations of the "same" entity (or taken to be the same). The concept "real number" is thought through different presentations, actual (Cauchy's sequences, Dedekind's cuts among others) or possible, but it must not be confused with anyone of them. In good cases, different presentations for the "same" entity are provably equivalent in the sense that the meaning of theorems valid in one case is preserved by theorems valid in the other case. The question of the "sameness" of referent through different presentations or definitions poses a difficult epistemological problem. Mathematics faces this problem constantly and solves it pragmatically by showing, in case it is possible, an equivalence relation between the entities under consideration.

For instance, topological spaces can be defined in many different ways, e.g., via open sets, via closed sets, via neighbourhoods (Hausdorff), via convergent filters, and via closure operations. These definitions describe "essentially the same" objects, what Category theory expresses via the notion of *concrete isomorphism*.

7. Probably the most fundamental action is thinking in terms of invariance; it operates in any mathematical area and corresponds to the task of isolating *intrinsic* or *stable* properties of the object under study. One wants indeed to

32. See [Marquis 2012, 9, fn 20]: "Each sort of abstract entity, for example, monoid, group, ring, field, topological space, partial order, etc., has its criterion of identity. It is certainly a nice feature of category theory that it provides a unified analysis of these criteria of identity as being isomorphisms in the appropriate category."

study not only the structure of some entity but also how it behaves under transformations. A few examples are below, taken from arithmetic, geometry, algebra, topology, algebraic topology, and category theory.

- The cardinal number of a set is invariant under the process of counting, angles are invariant under scalings, rotations, translations and reflections; for any circle the ratio of the circumference to the diameter is invariant and equal to π.
- Felix Klein characterized a geometry by a set of geometric invariants under a given group of symmetries; e.g., lengths, angles and areas are preserved with respect to the Euclidean group $E(n)$ of isometries (i.e., reflections, rotations, translations and combinations of these basic operations), while only the incidence structure and the cross-ratio are preserved under the most general projective transformations.
- Sylvester law of inertia: certain properties of the coefficient matrix of a real quadratic form (homogeneous polynomial of degree 2 in a number n of variables) remain invariant under a change of coordinates. Expressed geometrically, the law of inertia says that all maximal subspaces on which the restriction of the quadratic form is positive definite (respectively, negative definite) have the same dimension.
- In Hilbert's invariants theory the finite basis theorem states that every ideal in the ring of multivariate polynomials over a Nœtherian ring is finitely generated (invariance combined with reduction to a basis). Translated into algebraic geometry that means that every algebraic set over a field can be described as the set of common roots to a finite number of polynomial equations.
- The normal subgroups of a certain group G are the subgroups of G invariant (stable) under the inner automorphisms of G.
- The dimension of a topological space is invariant under homeomorphism.
- Algebraic invariants are used for classifying topological spaces up to homeomorphism or, more usually, to homotopy equivalence:[33] given two spaces X and Y, we say they are homotopy-equivalent or of the same homotopy type if there exist continuous maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ is homotopic to the identity map $\mathrm{id}X$ and $f \circ g$ is homotopic to $\mathrm{id}Y$.

  Going further, one defines the homotopy category as the category whose *objects* are topological spaces, and whose morphisms are homotopy *equivalence classes* of continuous maps. Two topological spaces X and Y are isomorphic in this category if and only if they are homotopy-equivalent. Then a functor on the category of topological spaces is

33. Two continuous functions from one topological space to another are homotopic iff one can be continuously deformed into the other, such a deformation being called a homotopy between the two functions.

homotopy-invariant if it can be expressed as a functor on the homotopy category.

These examples show that the ideas of functional relation, invariance, equivalence, classification, factorization and some others are working together and are using tools from one area (e.g., arithmetic and algebraic tools respectively) for characterizing entities belonging to another area (e.g., algebraic number theory and topological spaces respectively).

5 Conclusion

Mathematical abstraction consists in various processes increasing knowledge; so I have considered it from its epistemological aspect rather than from its logical or ontological aspect. The question whether the abstraction process is logical or psychological gives rise to argument. I think the process has evidently a logical side and a psychological side, the latter being by now very much investigated by cognitive scientists and neuroscientists. From the point of view of mathematical practice, abstraction is an indispensable tool of work and production. I have been interested here by the multiple ways of constructing and developing mathematical abstract objects, and I have tried to show which permanent actions are involved in all those ways.

Abstraction is very often linked with generalization; nevertheless there are abstract and non general objects, such as Dedekind's integers, which are a particular model of a general structure, and there are concepts that are equally abstract but have a different degree of generality: e.g., the concept of group is as abstract as the concept of field and it is more general. I had personally no example of a general procedure or entity, which would not involve abstraction at some level. J.-P. Marquis gives the example of passing from the notion of continuity of a function $f : \mathbf{R} \to \mathbf{R}$ at a point to that of continuity over a real interval [Marquis forthcoming, 17]. That leads me to think that generalization can sometimes be made without using abstracting processes, while any process of abstraction involves generalization.

Mathematical abstraction has more than one way; it is not limited to Aristotelian concept formation even though conceptualization, that is to say forming concepts by various procedures, is one essential way and is very characteristic of modern mathematics. Moreover, different ways are *simultaneously* used in constructions of higher and higher levels.

Some ways are known from the beginnings: idealization (geometrical shapes), invariance (invariant ratio between lengths or integers), factorizing (integers).

Other ways are more specific of modern mathematics:

- making a whole from an infinite number of unspecified elements,

- manipulating symbols, formulae, diagrams, sets of axioms as *being*, rather than *expressing* mathematical objects,
- setting up analogies between apparently different objects, sets, structures, theorems, etc., and correlatively dealing with *classes* of structures and theorems,
- considering functional relations or correspondences between elements, structures, functors,
- thematizing:
  - viewing operations of one level as objects of the successor level,
  - dealing with abstract structures and proving structure theorems with the help of structure-preserving maps,
  - considering equivalent classes of elements, of structures, of morphisms, etc., and proving classification theorems, transferring theorems between categories, etc.

The variety, wide enough, of the examples I have recalled shows that the notion of mathematical abstraction is plural and flexible. The abstraction process is open: new steps towards higher levels yielding more abstract, more sophisticated, and more encompassing concepts are to be expected.

Bibliography

ALEXANDROFF, Pavel [1935], Emmy Nœther's obituary, in: *Gesammelte Abhandlungen*, edited by N. Jacobson, Springer-Verlag, 1–11, 1983.

ARISTOTLE [1933], *Métaphysique* μ, Paris: Vrin, French translation J. Tricot.

—— [1962], *Seconds Analytiques*, Paris: Vrin, French translation J. Tricot.

ARTIN, Emil [1962], Zur Problemlage der Mathematik, in: *The Collected Papers*, edited by S. Lang & J. Tate, Reading, Mass.: Addison-Wesley.

AWODEY, Steve [1996], Structure in mathematics and logic: A categorical perspective, *Philosophia Mathematica*, 4(3), 209–237.

BENACERRAF, Paul [1973], Mathematical truth, *The Journal of Philosophy*, 70(19), 661–679, reprint in: Benacerraf, P. and Putnam, H., *Philosophy of Mathematics. Selected Readings*, 2nd edn, Cambridge University Press, 1983, 403–420.

BENIS SINACEUR, Hourya [1991], *Corps et Modèles*, Mathesis, Paris: Vrin, 2nd edn., 1999.

—— [2000], The nature of progress in mathematics: The significance of analogy, in: *The Growth of Mathematical Knowledge*, edited by E. Grosholz & H. Berger, Boston: Kluwer Academic Publishers, *Synthese Library*, vol. 289, 281–293.

—— [2008], *Richard Dedekind, la création des nombres*, Mathesis, Paris: Vrin.

Bolzano, Bernard [1851], *Die Paradoxien des Unendlichen*, Leipzig: Felix Meiner Verlag, Hamburg, 1975; French translation by H. Benis Sinaceur, Éditions du Seuil, 1993.

Boolos, George [1998], *Logic, Logic, and Logic*, Cambridge, Mass.: Harvard University Press.

Brouwer, Luitzen E. J. [1909], *Het wezen der Meetkunde*, Amsterdam: Clausen, French translation in Largeault, J. *Intuitionisme et théorie de la démonstration*, Paris: Vrin, Mathesis, 1992, 25–37.

Burgess, John P. & Rosen, Gideon [1997], *A Subject with no Object*, Oxford: Oxford University Press.

Cleary, John J. [1985], On the terminology of 'abstraction' in Aristotle, *Phronesis*, 30(1), 13–45.

Dedekind, Richard [1855-1858], Aus den Gruppen-Studien, in: *Gesammelte mathematische Werke*, Braunschweig: Vieweg, vol. III, 439–445, 1932.

—— [1888], Was sind und was sollen die Zahlen?, in: *Gesammelte Mathematische Werke*, edited by R. Fricke, E. Noether, & O. Ore, Braunschweig: Vieweg, vol. III, 335–391, trad. fr. [Benis Sinaceur 2008, 131–220].

Dehaene, Stanislas [2011], *The Number Sense: How the Mind Creates Mathematics*, New York: Oxford University Press.

Frege, Gottlob [1884], *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Breslau: Wilhelm Koebner, French translation by Cl. Imbert, *Les Fondements de l'arithmétique : recherche logico-mathématique sur le concept de nombre*, Paris: Éditions du Seuil, 1969.

Gentner, Dedre [1983], Structure-mapping: A theoretical framework for analogy, *Cognitive Science*, 7, 155–170.

Hasse, Helmut [1931], Über p-adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperklomplexer Zahlsysteme, *Mathematische Annalen*, 104, 495–534.

Hausdorff, Felix [1914], *Grundzüge der Mengenlehre*, Leipzig: Von Veit.

HEGEL, Georg Wilhelm Friederich [1812-1816], *Wissenschaft der Logik, I-III*, Nürnberg: J. L. Schrag, new edition by G. Lasson, Leipzig: Felix Meiner Verlag, 1951. URL: http://www.deutschestextarchiv.de/book/view/hegel_logik02_1816?p=9.

HILBERT, David & BERNAYS, Paul [1934-1939], *Grundlagen der Mathematik*, Springer-Verlag, 2nd edn., 1968-1970.

KANT, Immanuel [1800], *Logik: ein Handbuch zu Vorlesungen*, Königsberg: Friedrich Nicolovius, English translation by John Richardson, London, 1819. URL: http://books.google.fr/books?id=2CFNp0S753sC&printsec=frontcover&hl=fr&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false.

KREISEL, Georg [1985], Mathematical logic: Tool and object lesson for science, *Synthese*, 62(2), 139–151.

KRÖMER, Ralf [2007], *Tool and Object: A History and Philosophy of Category Theory*, Birkhäuser.

LANDRY, Elaine & MARQUIS, Jean-Pierre [2005], Categories in context : Historical, foundational, and philosophical, *Philosophia Mathematica*, 13(1), 1–43.

LEWIS, David [1986], *On the Plurality of Worlds*, Oxford: Basil Blackwell.

LOCKE, John [1689], *An Essay Concerning Human Understanding*, vol. III. URL: http://www.gutenberg.org/ebooks/10615.

MARQUIS, Jean-Pierre [2012], Categorical foundations of mathematics, or how to provide foundations for abstract mathematics, *The Review of Symbolic Logic*, 6(1), 1–25.

—— [forthcoming], Mathematical abstraction, conceptual variation and identity, Draft Online.

PARSONS, Charles [2008], *Mathematical Thought and its Objects*, New York: Cambridge University Press.

POINCARÉ, Henri [1900], Du rôle de l'intuition et de la logique en mathématiques, in: *Compte rendu du deuxième Congrès International des Mathématiciens*, Paris: Gauthier-Villars, 115–130, 1902.

—— [1908], L'avenir des mathématiques, in: *Atti del IV Congresso Internazionale dei Matematici (Bologna)*, Roma: Academia dei Lincei, 167–182, 1909.

—— [1928], *Œuvres I*, Paris: Gauthier-Villars.

Quine, Willard v. Orman [1951], Two dogmas of empiricism, *Philosophical Review*, 60, 20–43, reprint in *From a Logical Point of View*, 2nd edn rev., Harper & Row, 1962, 20–46.

—— [1960], *Word and Object*, New York: John Wiley & Sons.

—— [1990], *Pursuit of Truth*, Cambridge, Mass.: Harvard University Press, French translation by M. Clavelin, *La Poursuite de la vérité*, Paris: Éditions du Seuil, 1993.

Riemann, Bernhard [1854], Über die Hypothesen welche der Geometrie zu Grunde liegen, in: *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, edited by R. Dedekind, vol. 13, 133–150, 1867. URL: `http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/`.

Rosen, Gideon [2012], Abstract objects, in: *Stanford Encyclopedia of Philosophy*, first published 2001.

Steinitz, Ernst [1910], Algebraische Theorie der Körper, *Journal für die reine und angewandte Mathematik*, 137, 167–309. URL: `http://eudml.org/doc/149323`.

Szczeciniarz, Jean-Jacques [1999], Abstraction, in: *Dictionnaire d'Histoire et de Philosophie des Sciences*, edited by D. Lecourt, Paris: Presses Universitaires de France, 4–9.

Tammet, Daniel [2005], A genius explains, *The Guardian*, 12 Feb., 34.

Weyl, Hermann [1921], Über die neue Grundlagenkrise der Mathematik, *Mathematische Zeitschrift*, 10, 37–79, English translation in Mancosu, P., *From Brouwer to Hilbert*, Oxford University Press, 1997, 86–118.

—— [1932], Topologie und abstrakte Algebra als zwei Wege des mathematischen Verständnisses, *Unterrichtsblätter für Mathematik und Naturwissenschaften*, 32, 117–188, reprint in *Gesammelte Abhandlungen III* (Hrsg. K. Chandrasekharan), Springer-Verlag, 1968, 348–358.

—— [1935], Emmy Nœther, *Scripta Mathematica*, 3, 201–220, reprint in *Gesammelte Abhandlungen III*, 1968, Berlin; New York: Springer-Verlag, 425–444.

—— [1951], A half century of mathematics, *American Mathematical Monthly*, 58(8), 523–553, reprint in *Gesammelte Abhandlungen IV*, 464–496. URL: `http://www.jstor.org/discover/10.2307/2306319?uid=3738016&uid=2129&uid=2134&uid=2&uid=70&uid=4&sid=21102906564633`.

Zalta, Edward [1983], *Abstract Objects. An Introduction to Axiomatic Metaphysics*, *Synthese Library*, vol. 160, Dordrecht: Reidel.

—— [2013], Frege's logic, theorem, and foundations for arithmetic, in: *Stanford Encyclopedia of Philosophy*, first published 1998.